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USE OF A TRANSCENDENTAL APPROXIMATION IN TRANSIENT CONDUCTION ANALYSIS

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INTRODUCTION

For many non-linear problems it is generally necessary to obtain solutions by either numerical or approximate methods. In a recent paper (1) it was noted that typical examples of these problems are those involving laminar boundary layer flows and non-linear heat conduction as investigated by Biot's variational method. In that paper use of a transcendental approximation in laminar boundary layer analysis was examined. In the present paper, use of the same transcendental approximation in non-linear heat conduction is discussed.

In previous studies of typical one-dimensional, transient heat conduction problems with non-linear boundary conditions (2, 3) the temperature distribution has been represented subject to a two-fold approximation. First, the boundary condition infinitely distant from the surface is brought to a finite distance from the surface, this distance being called the penetration depth; then the temperature is approximated by a polynomial in the region from the surface to the penetration depth, being regarded as constant at greater depths. For the cases considered previously, it is possible to obtain quickly at least the asymptotic solutions for short and for long times. However, it is not clear that the solutions will be roughly of equal accuracy, since the polynomial profile used may be less suited to represent the true profile at one of the (asymptotic) times than at the other. The profile function used here is adaptable to the extent that the profile shapes obtained for each asymptotic case need not be the same. Further, it is known that the actual profile must be some form of transcendental function, and a transcendental approximation can have a closeness of indefinitely high order to the exact solution.

In this paper a particular example of a transient conduction problem with a non-linear boundary condition is considered, for which other solutions have been obtained. This example consists of unsteady one-dimensional conduction in a semi-infinite slab with the boundary condition that the heat flux at the surface is proportional to the n-th power of the surface temperature and with the initial condition that the slab temperature is uniform, forming the reference zero, i.e. the zero of an appropriate empirical temperature scale.

Solutions for this form of boundary condition are useful for a variety of problems. One practical problem for which they have been used is transient conduction due to unsteady radiation in an enclosure.

NOTATION

A	coefficient
a	profile parameter, function of n
B	coefficient
b	profile parameter, function of n
C	coefficient
c	specific heat
D	dissipation function; also, a coefficient
$-Ei(-u)$	exponential integral
$E_p(u)$	general exponential integral
F	heat flux at $x = 0$
f	coefficient
g_k	constants
$\utilde{H}$	heat flow field; $\dot{\utilde{H}}$, heat flux field
I	a specific integral
$i^m\mathrm{erfc}\ \eta$	m-th repeated integral of the error function
J	coefficient
K	coefficient
k	thermal conductivity; also, running index
M	function of u_o
M_m	coefficients
m	running index
n	exponent
$\utilde{n}$	unit normal vector
p	running index
Q_i	thermal force
q_i	i-th generalized coordinate; q_1, surface temperature; q_2, penetration depth
S	surface
T	arbitrary constant temperature
t	time
u	$\exp\ (a + b\eta)$
u_o	$\exp\ a$
V	a thermodynamic potential
v	volume

x, y, z	space coordinates
pro η	dimensionless temperature profile approximation
α_o, β_o, γ_o	coefficients
$\Gamma(1+n)$	gamma function
γ	Euler's constant
ε	an arbitrary small number
ζ	dimensionless space variable
η	general variable
θ	temperature
ρ	density
σ	scaling factor
τ	dimensionless time
χ	dimensionless penetration depth
ψ	dimensionless surface temperature, $x = 0$

Biot's Variational Method for Transient Conduction

A variational principle for analysis of heat conduction has been described extensively by Biot (t, 5), who has also given some examples of its use. An alternative, complementary method for deriving the principle was described by Boley & Weiner (6). In essence, it is supposed that one can consider a heat flow vector field $\underset{\sim}{H}$ to exist within a body, such that the time-rate of change $\dot{\underset{\sim}{H}}$ is the heat flux across an area normal to $\dot{\underset{\sim}{H}}$. From the First Law, in the absence of work, one has that

$$c\rho\theta = -\text{div}\ \underset{\sim}{H} \tag{1}$$

In particular, if the heat flow field can be expressed as a function of n "generalized coordinates" $q_i\ (t)$, so that

$$\underset{\sim}{H} = \underset{\sim}{H}\ (q_i;\ x,\ y,\ z,\ t) \tag{2}$$

then it is possible to write a variational principle in the form

$$\frac{\partial V}{\partial q_i} + \frac{\partial V}{\partial \dot{q}_i} = Q_i \tag{3}$$

where

$$Q_i = \int_S \theta\ \underset{\sim}{n}\cdot(\partial\underset{\sim}{H}/\partial q_i)\ dS \tag{4}$$

and, common to the n equations [3],

$$V = \int (1/2)c\rho\ \theta^2\ dv \tag{5}$$

$$D = \int \frac{1}{2k}\ \dot{H}^2\ dv\ . \tag{6}$$

This variational principle is equivalent to the heat conduction equation in an isotropic medium. When the temperature field is one-dimensional in space, it and the heat flow field are related simply. In order to utilize the method, some particular form of temperature profile is assumed, and a sufficient number of generalized coordinates assigned to describe

it. From eqn. [1] the heat flow field corresponding to the assumed profile can be determined, and subsequently the dissipation function D and the potential V.

As an alternative to determining the n generalized coordinates from eqn. [3], it is possible to determine one through use of a compatibility condition on the surface heat flux (e.g., by forcing H_o for a semi-infinite body to satisfy exactly the heat flux boundary condition), leaving $(n - 1)$ coordinates to be determined from the variational principle.

The Profile Function

The function for profile approximation introduced by Richardson (7) is

$$\text{pro}\ \eta = \exp\,[-\exp\,(a + b\eta)]\ /\ \exp\,[-\exp\, a] \qquad [7]$$

and this function satisfies the conditions

$$\text{pro}\ \eta = 1 \quad \text{at} \quad \eta = 0$$
$$\text{pro}\ \eta \rightarrow 0 \quad \text{as} \quad \eta \rightarrow \infty\ .$$

This profile function is a two-parameter profile, and in general two independent equations are required in order to determine the relevant parameters.

The Differential Equations for the Profile Function

The equations found by Biot's method depend upon the specific temperature profile which has been assumed. It is convenient to write

$$\eta = x/q_2\ ,$$
$$u = \exp\,(a + b\eta)\ ,$$

and $\qquad u_o = \exp\, a\ ,$

so that the profile assumed is

$$\theta = q_1 \exp(-u) / \exp(-u_o) . \qquad [8]$$

It is also assumed that the heat flux boundary condition F belongs to the class of functions

$$F = f\theta_o{}^n = f\, q_1{}^n . \qquad [9]$$

It is convenient to form dimensionless groups for q_1, q_2 and t, but it is found that the parameters of the problem do not provide enough quantities and it is necessary to introduce an arbitrary temperature or length (which subsequently cancels out in the solutions). If the former is chosen and denoted by T, the following dimensionless groups can be established:

$$\begin{aligned} \psi &= q_1 / T \\ \chi &= q_2 f\, T^{n-1} / k \\ \tau &= t\, f^2\, T^{2(n-1)} / kc\rho \end{aligned} \qquad [10]$$

The heat flow field is given by

$$H = \frac{c\rho q_1 q_2}{b \exp(-u_o)} \mathrm{Ei}(-u) , \qquad [11]$$

so that by determining eqn. [3] for q_1 it can be shown that

$$J\psi + (3/2)A\psi\chi\dot{\chi} + A\dot{\psi}\chi^2 = 0 \qquad [12]$$

while the surface flux compatibility condition gives that

$$\dot{\psi}\chi + \psi\dot{\chi} = K\psi^n . \qquad [13]$$

These equations have the same form as those obtained by Richardson (3), but the coefficients differ because of the different assumed profile. The coefficients which arise here are

$$A = \frac{1}{b^2}\int_0^\infty \frac{[Ei\ (-u)]^2}{\exp\ (-2u_o)}\ d\eta \qquad [14]$$

$$B = \frac{1}{b}\int_0^\infty \frac{\eta \exp\ (-u)\ Ei\ (-u)}{\exp\ (-2u_o)}\ d\eta \qquad [15]$$

$$C = -\frac{Ei\ (-2u_o)}{b\ \exp\ (-2u_o)}$$

$$D = -\frac{Ei\ (-u_o)}{b\ \exp\ (-u_o)} \qquad [16]$$

$$J = C - D$$

$$K = 1/D\ . \qquad [17]$$

In deriving eqn. [12], advantage is taken of the fact that $B = A/2$, which can be shown by noting that $\eta = (\ln u - a)/b$, transforming to u as the variable of integration, and integrating by parts.

Equations [12] and [13] can be solved easily for the cases $n = 0$ (for which there is also an exact solution) and $0 < n < 1$. The solution for $n = 0$ (i.e. constant heat flux) is

$$\chi = \beta_o \tau^{1/2} = -\frac{4J}{5A}\ \tau^{1/2}$$

$$\psi = \alpha_o \tau^{1/2} = K - \frac{5A}{4J}\ \tau^{1/2} \qquad [18]$$

and for $0 < n < 1$ the solution is

$$\chi = \gamma_o \tau^{1/2} = -\frac{J}{A}\ \frac{60}{45 + \frac{30}{1-n}}\ \tau^{1/2}$$

$$\psi = \frac{\gamma_o}{K}\ \frac{(2 - n)}{2(1 - n)}^{1/(n - 1)}\ \tau^{1/2(1 - n)} \qquad [19]$$

It is to be noted that this solution cannot be used near $n = 1$. (In (3) these sets of solutions were incorrectly indicated to be asymptotic solutions instead of complete solutions.)

Evaluation of Coefficients

The next step in completing the solutions is to evaluate the coefficients A, J and K. These coefficients can be obtained as functions of a and b. Finally a criterion can be established for obtaining a and b, thereby completing the solutions.

It may be noted that Ab^3, Jb and K/b are each functions of u_o only, and therefore of a only. It can be seen from their definitions that Jb and K/b can be obtained from tables of exponential integrals using straightforward manipulations. The integral for Ab^3 cannot be evaluated directly, but the integral can be expanded in an infinite series which can be integrated term by term. It was found that the solutions required use of values of u_o roughly in the range of 3.0 to 8.0. For small values of the argument the exponential integral can be written

$$Ei(-u) = \gamma + \ln(u) - u + (1/2!2)u^2 - (1/3!3)u^3 + \ldots \qquad [20]$$

This series converges, but for values of $u > 1$ the terms initially increase in magnitude, making it necessary to take many terms to obtain accurate results. For large values of u

$$Ei(-u) = e^{-u}\{u^{-1} - u^{-2} + 2u^{-3} - (3!)u^{-4} + \ldots\} \qquad [21]$$

This second series is asymptotic, the terms initially converging and then diverging.

It can be seen that

$$A = \frac{\exp(2u_o)}{b^3}\int_{u_o}^{\infty}\{Ei(-u)\}^2\,\frac{du}{u}$$

$$= \frac{\exp(2u_o)}{b^3}\, I \text{ , say.} \qquad [22]$$

For values of $u_o \geq 4$ the asymptotic expansion [21] was substituted, giving

$$I = \int_{u_o}^{\infty} e^{-2u} \{ \sum_{m=1}^{\infty} (-1)^{m+1} (m-1)! u^{-m} \}^2 \frac{du}{u}$$

$$= - \int_{u_o}^{\infty} e^{-2u} \sum_{k=3}^{\infty} (-1)^k g_k u^{-k} du$$

$$= - \sum_{k=3}^{\infty} (-1)^k g_k \int_{u_o} u^{-k} \exp(-2u) du , \qquad [23]$$

where values of g_k are $g_3 = 1$, $g_4 = -2$, $g_5 = 5$, $g_6 = -16$, $g_7 = 64$, $g_8 = -312$, etc.

Now, the general exponential integral is defined (8) as

$$E_p(u) = \int_1^{\infty} x^{-p} \exp(-xu) dx , \qquad [24]$$

so that the typical integral term of I is

$$\int_{u_o}^{\infty} u^{-k} \exp(-2u) du = u_o^{1-k} E_k(2u_o)$$

whence

$$I = -u_o \sum_{k=3}^{\infty} (-1)^k g_k u_o^{-k} E_k (2u_o)$$

so that the coefficient A could be evaluated as a series of general exponential integrals

$$A = - \frac{u_o \exp(2u_o)}{b^3} \sum_{k=3}^{\infty} (-1)^k g_k u_o^{-k} E_k (2u_o) . \qquad [25]$$

In computation the series was summed to the smallest term; if the consequent maximum error was considered too large, Euler's transformation was applied, and further terms taken until the maximum error was reduced to a satisfactory magnitude.

Determination of Constants for the Profile Function

The coefficients A, J and K of the differential equations were determined as functions of u_o and b in the last section. The values

of u_o can be found by application of the heat flux boundary condition, eqn. [9]:

$$F = f\, \theta_o^n = f\, q_1^n = -k(\partial\theta/\partial x)_o .$$

Differentiation of eqn. [8] gives

$$F = k\, q_1\, b\, u_o \,/\, q_2 ,$$

which leads to

$$\psi^{n-1}\, \chi \,/\, b\, u_o = 1 .$$

Substitution of the solutions [19] gives

$$b\, u_o\, A\, K \,/\, J = M = 2(2 - n)/(3n - 5) . \qquad [26]$$

The left hand side of eqn. [26] is a function of u_o only, but the value of u_o corresponding to a specific value of n cannot be extracted directly. Values of M were computed for values of $u_o = 3.0\ (1.0)\ 8.0$ with use of tables and the asymptotic expansion, eqn. [21], and for values of $u_o \leq 4.0$ by integration using eqn. [20]. In calculation at $u_o = 4$, one of the series used in determining A had to be taken to twenty-five terms to evaluate the integral to eight significant digits. Checks were made by overlapping and use of alternative series. In order to determine values of u_o for specified n, it was found convenient and adequate to use linear interpolation between values of M^4 vs. $\log u_o$. It was estimated that interpolation errors would correspond to less than $\pm\, 0.001\, M$. Once values of u_o have been found, values of a can be determined immediately. The values of b remain to be found.

For the polynomial approximation of the temperature distribution the penetration depth is defined as the point where $\theta = 0$, i.e. $x = q_2$. With the profile function $\theta \to 0$ as $x \to \infty$ and hence q_2 must be defined differently. It is convenient to put $x = q_2$ where $\theta/q_1 = \varepsilon$, where ε is an arbitrary, very small number. From the profile function it follows that

$$b = \ln(1 - e^{-a} \ln \varepsilon) \ . \qquad [27]$$

It was decided to use $\varepsilon = 0.01$ here. It must be emphasized that the value of b depends upon the value of ε. With this convention, it is not significant to compare directly the coefficients of the penetration depth solutions from use of polynomial and transcendental profiles.

Results and Accuracy of Computations

Constant-flux solution The values found for the constant-flux solution were $u_o = 2.95$,

$$\psi = 1.130\ \tau^{1/2}$$

$$\chi = 3.18\ \tau^{1/2} \qquad [28]$$

which can be compared with Richardson's (3) parabolic approximation

$$\psi = 1.157\ \tau^{1/2}\ ; \qquad \chi = 2.59\ \tau^{1/2}$$

and with the exact solution

$$\psi = 1.1284\ \tau^{1/2}\ .$$

It can be seen that the coefficient for $\psi/\tau^{1/2}$ using the profile function differs from the exact solution by about 0.2 per cent and from the parabolic approximation by about 2.4 per cent.

Solutions for $n > 0$ A set of solutions was computed for a range of n where the computations did not become inordinately long. In Table I values of a and b are provided, and in Table II values of the surface temperature coefficients $\psi/\tau^{1/2(1-n)}$ (parabolic profile), $\psi/\tau^{1/2(1-n)}$ (transcendental profile) and penetration depth coefficient (transcendental profile) are listed, together with the percentage differences of the surface temperature coefficients.

For the sake of comparison, two temperature profiles have been plotted

in Fig. 1. The scale of slab depth η has been adjusted by using scaling factors σ so that both profiles have the same slope at $\eta = 0$. It can be seen that the profile for the constant-flux solution has a somewhat different shape from that for the solution with $n = 0.7$. Profiles for intermediate values of n fall between those shown.

Another comparison has been made in Fig. 2 for the constant-flux solution. The ordinate is of temperature, normalized such that the temperature on the axis (which corresponds to the body surface) is always unity; the abscissa, the slab depth, has been adjusted such that all temperature profiles shown have a slope of unity at the body surface. The middle solid line represents the profile of the analytic solution of the constant flux case; two other solid lines represent the parabolic and the simple exponential approximations respectively. The dashed line represents the profile approximation used here. Since this lies so close to (and crosses) the analytic solution, only parts can be shown without obscuring the analytic solution. This figure demonstrates well that the profile function used here is a closer approximation than are the others cited.

The computations using the exponential integrals were based upon the tables of Pagurova (8), with values for integrals beyond the range of the tables generated using relations given in the same reference. Computations for other quantities described here were made using the handbook of Abramowich and Stegun (9), with checks being made in relevant references given therein. The accuracy of the computations varies with n in the solutions, but it is believed to be not worse than 0.1 per cent in any case.

Analytic Solutions for $n = 1.0$

The analytic solution for $n = 0$, i.e. constant heat flux, is well known. The problem can also have an analytic solution at $n = 1.0$ if the initial condition described in the Introduction is slightly altered so that the slab temperature is uniform but above the reference zero by an arbitrary amount. Without this alteration the solution recovered is the trivial case $\theta(x, t) = 0$. The arbitrary uniform temperature can be chosen as the unit temperature of the θ temperature scale; the solution

is proportional to it. The solution can be written

$$\theta(x, t) - 1 = \sum_{m=0}^{\infty} M_{m+1} \tau^{(m+1)/2} \frac{i^{m+1}\mathrm{erfc}(\zeta/\tau^{1/2})}{i^{m+1}\mathrm{erfc}(0)} \tag{[29]}$$

with $M_1 = \frac{2}{\sqrt{\pi}}$

$$M_{m+1} = \frac{\Gamma(\frac{n}{2} + 1)}{\Gamma(\frac{n+1}{2} + 1)} M_m$$

$$\tau = \frac{tf^2}{kc\rho}, \quad \zeta = \frac{xf}{2k}$$

(where f has been normalized with respect to the initial temperature). The function i^m erfcη is the m-th repeated integral of the error function, for which useful tables, recurrence relations, expansions and so forth exist. At small times, the term for $m = 0$ is dominant and this coincides with the constant flux solution discussed previously. For longer times, the series provides a shifting average of the repeated integrals of the error function. Successive integrals have profile shapes which move from the central profile of Fig. 2 towards the simple exponential. The approximate profiles found here for $0 < n < 1.0$ have a trend in the same direction. For very long times, the asymptotic behavior of i^m erfcη as $m \to \infty$ is important. This can be determined from the series expansion

$$i^m \mathrm{erfc}\eta = \sum_{p=0}^{\infty} \frac{(-1)^p \eta^p}{2^{m-p}\, p!\, \Gamma(1 + \frac{m-p}{2})}$$

in which the terms corresponding to $p = m+2, m+4, m+6, \ldots$ are understood to be zero. As m tends to infinity, it can be shown that the normalized repeated integral of the error function (i^m erfcη/i^m erfc0) is given asymptotically by a simple negative exponential function of η.

Discussion

It can be seen from Fig. 1 that the solution of the problem does take advantage, so to speak, of the flexibility of the transcendental

profile by finding that different shapes are appropriate to different conditions. This adaptive feature of the transcendental profile should assist in providing results which are more accurate than with fixed simple profiles. This hope is borne out clearly in the constant-flux solution, where the accuracy of the surface temperature coefficient is improved by a tenfold order of magnitude compared with the simple polynomial and exponential profiles listed in Table I of Lardner's paper (2). It is also demonstrated by Fig. 2.

For $0 < n < 1.0$ no exact solutions exist with which the results can be compared. The calculations become increasingly difficult as n approaches unity. This is due partly to the need to generate exponential integrals beyond the range of available tables, and partly to the behavior of the profile function. In the profile function the parameter a essentially specifies the shape of the profile, while the parameter b specifies the extent of the profile. Thus, two profiles drawn with the same values of a and different values of b have the same shape, but not vice versa. As $a \to \infty$, $\text{pro}\eta \to \exp[-(b \exp a)\eta]$. Over the major range of $\text{pro}\eta$, this limit is approached rapidly. Even with $a = 5.0$, $\text{pro}\eta$ is close to its limit. This means that if an attempt is made to fit the profile function to a function which is close to the simple exponential, the determination of a becomes extremely insensitive; large changes in a produce small changes in the profile shape. It is increasingly difficult to obtain values of a to a specified number of significant digits.

It is noteworthy that the difference between the surface temperature coefficients listed in Table II increases smoothly from the value of 2.4% at $n = 0$ to 12% at $n = 0.73$, the difference increasing roughly exponentially with n. The difference between surface temperature coefficients is sufficiently large to be significant in application.

These differences occur because the transcendental profile changes its shape with n, which the parabolic profile cannot do. It is very probable that the solutions with the transcendental profile are more accurate, but at present this cannot be demonstrated directly. However, it is possible to make comparisons with the analytical solutions for $n = 0$ and $n = 1.0$ and show that the temperature profile shape generated here for $0 < n < 1.0$ varies smoothly between the analytic limits. The significant feature of the results upon which a comparison can be based is the

parameter a, which determines the shape of the profile. It has been noted above that the asymptotic behavior of the analytic solution for $n = 1.0$ is that the profile becomes a simple exponential; in approximating this shape a becomes large. For $n = 0$, the short-time solution is valid at all times, so that in this instance it is also the long-time solution. The values of a for solutions in the range $0 < n < 1.0$ are shown in Fig. 3. In this it can be seen that the values found here fit well between the analytic limits. This feature is absent in all fixed profiles, such as polynomials. The comparison provides conducive evidence that the solutions presented here are considerably more accurate than the corresponding solutions available previously.

Summary

(1) Biot's variational method is applied to a problem of transient heat conduction in a semi-infinite slab subject to a non-linear boundary condition.

(2) A two-parameter transcendental approximation is used for the temperature profile. This approximation has the advantage that its shape is not fixed, so that the profile determined for each case considered can have the shape most appropriate to it.

(3) Computations for solutions utilizing the general exponential integral are described, and two exact solutions are mentioned for cases which bound the examples computed.

(4) Comparison of the variational solution using the transcendental approximation with the exact solution for the limiting case of constant heat flux demonstrates an error of less than 0.2 per cent in the surface temperature coefficient and very close representation of the true temperature profile. This corresponds to a very considerable improvement over solutions obtained previously with other profile approximations.

(5) Comparison of the variational solution for other examples shows that the computed profile shapes have a uniform variation which is in the correct direction to merge with the limiting case of surface heat flux directly proportional to surface temperature.

(6) It is concluded that the solutions obtained here using the transcendental approximation are considerably more accurate than those previously available.

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Table I

Solutions for $0 < n < 1$

n	a	b
0.1	1.13287	0.90959
0.2	1.21696	0.86023
0.3	1.31709	0.80370
0.4	1.43553	0.73998
0.5	1.57704	0.66855
0.6	1.75592	0.58528
0.7	1.98445	0.49042
0.73	2.06623	0.45951

Table II

n	Surface temperature coefficients $\psi/\tau^{1/2(1-n)}$ parabolic profile	Surface temperature coefficients $\psi/\tau^{1/2(1-n)}$ transcendental profile	Penetration depth $\chi/\tau^{1/2}$ transcendental profile	Difference in surface temperature coefficients (per cent)
0.1	1.1347	1.1053	3.09136	2.7
0.2	1.0996	1.0658	3.05915	3.2
0.3	1.0453	1.0064	3.01486	3.9
0.4	0.96170	0.91769	2.96901	4.8
0.5	0.83340	0.78593	2.92637	6.0
0.6	0.64006	0.59309	2.76203	7.9
0.7	0.37095	0.33493	2.57070	10.7
0.73	0.28085	0.25068	2.49858	12.0

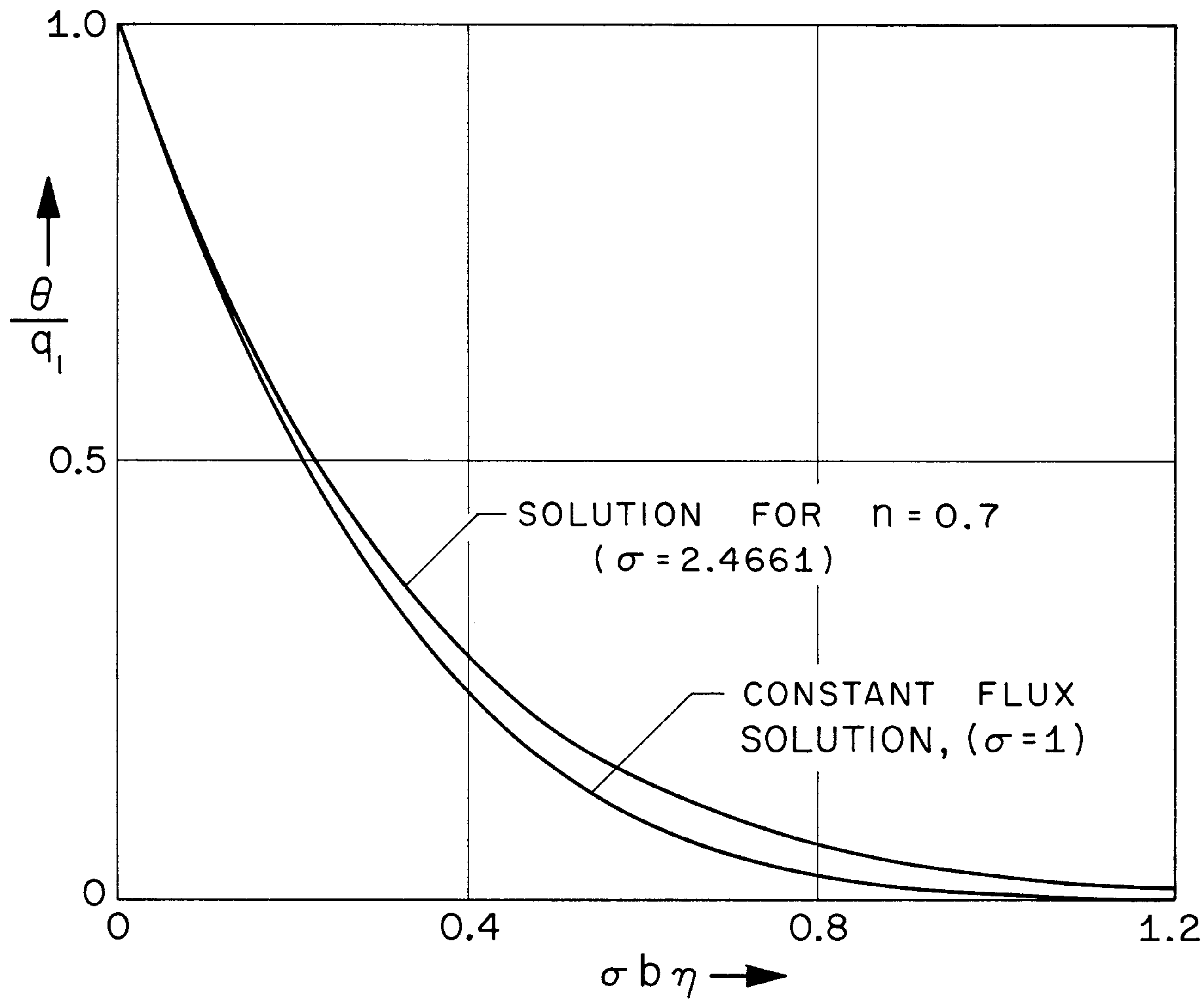

Figure 1.- Comparison of the shapes of temperature profiles. The profiles have been drawn with different scale factors to make the initial slopes identical.

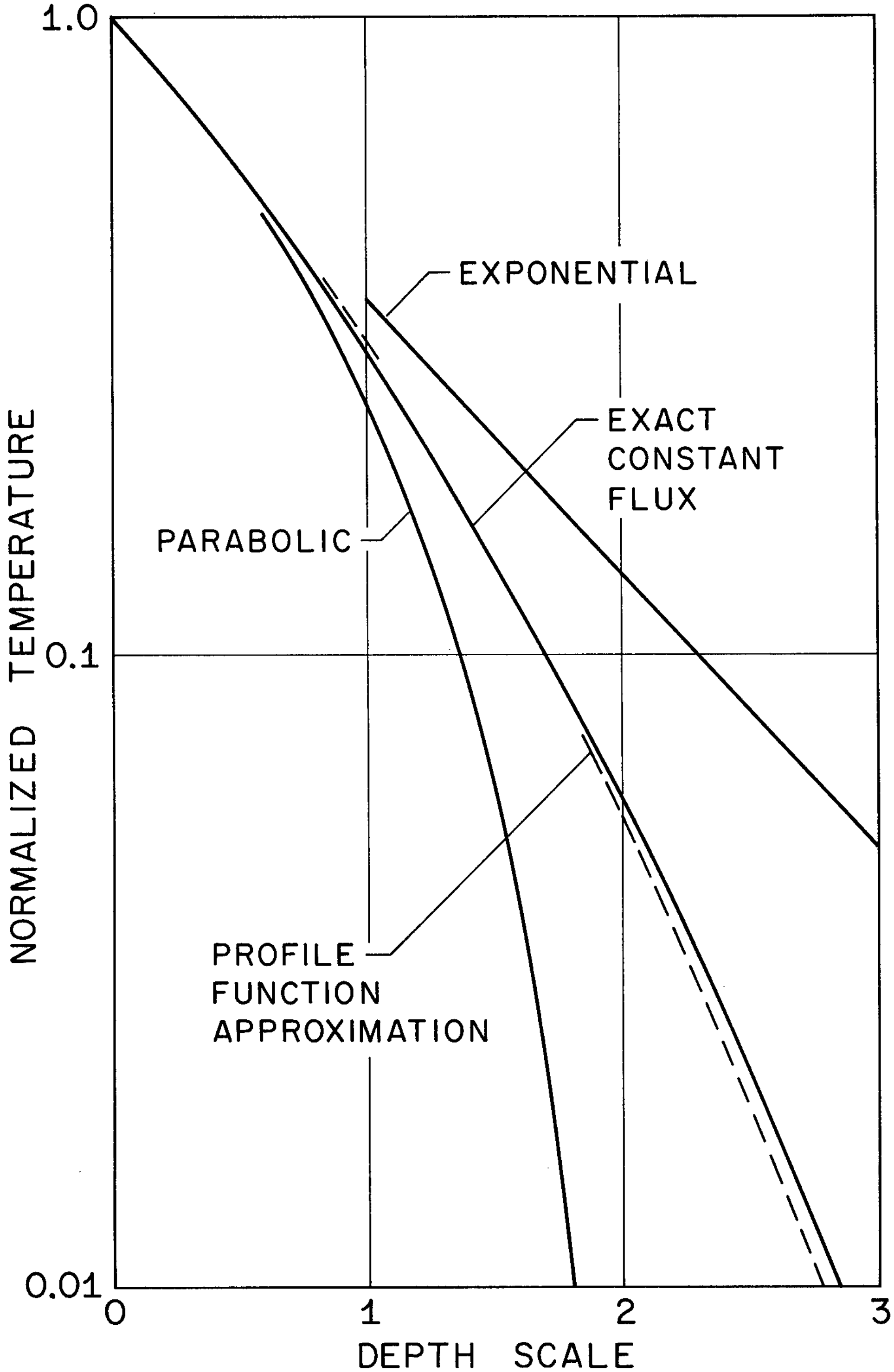

Figure 2.- Comparison of the exact temperature profile for the constant flux solutions with various approximations. The approximation function used here (dashed curve) gives a very close approximation to the exact profile.

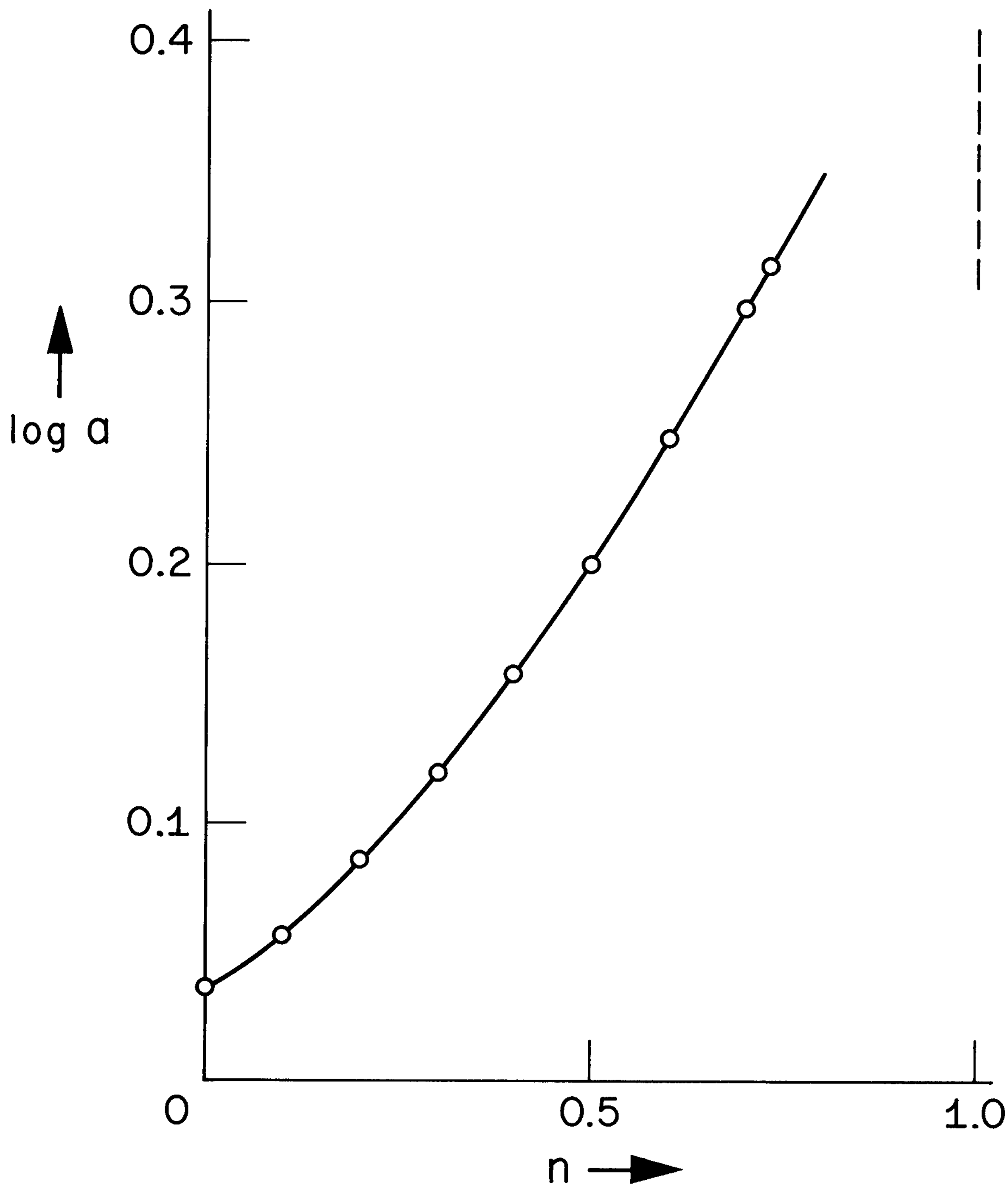

Figure 3.- Comparison of the profile shape parameter for solutions as a function of n. At $n = 1$ the exact solution corresponds to $a \to \infty$, but even $\log a = 0.70$ gives a very close approximation.

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